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Asymptotic Chebyshev Centers

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1. INTRODUCTION

Let h be the Hausdorff metric on the space $B(X)$ of all nonempty bounded closed subsets of a normed linear space X. We recall that

$$h(A, B) = \max\{\sup_{a \in A} \operatorname{dist}(a, B), \sup_{b \in B} \operatorname{dist}(b, A)\}$$

for any $A, B \in B(X)$, where

$$\operatorname{dist}(a, B) = \inf_{b \in B} \|a - b\|.$$

Denote by $B_\infty(X)$ the metric space of all sequences $\{C_n\}$ of subsets $C_n \in B(X)$ such that the union $\bigcup C_n$ is a bounded subset of X, endowed with the metric

$$H(C, D) = \sup_n h(C_n, D_n); \qquad C = \{C_n\}, D = \{D_n\} \in B_\infty(X).$$

For any sequence $C = \{C_n\} \in B_\infty(X)$, define the functional f_C on X by the formula

$$f_C(x) = \limsup_{n \to \infty} \sup_{z \in C_n} \|x - z\|. \tag{1.1}$$

If M is a nonempty closed convex subset of X, then an element $x_C \in M$ such that

$$f_C(x_C) = \inf_{x \in M} f_C(x) \tag{1.2}$$

is called the *asymptotic Chebyshev center* of the sequence $C = \{C_n\} \in B_\infty(X)$ with respect to M. The set (perhaps empty) of all such elements x_C is

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denoted by $P_M(C)$. Note that the notion of asymptotic Chebyshev center x_C includes the following fundamental notions from theories of approximation and fixed points:

(i) the *best approximation* x_C in M to an element $z \in X$, in the case when $C_n = z$ for every n;

(ii) the *Chebyshev center* x_C of a set $A \in B(X)$, whenever $M = X$ and $C_n = A$ for all n [1, 8];

(iii) the *relative center* x_C of a set $A \in B(X)$ with respect to M, if $C_n = A$ for every n [3, 7];

(iv) the *asymptotic center* x_C of a bounded sequence $\{x_n\}$ in M, whenever $C_n = x_n$ for all n [6, 9];

(v) the *asymptotic center* x_C *for a net* $C = \{C_n\} \in B_\infty(X)$, where $C_n \supset C_{n+1}$ for all n [2, 10, 11].

In this paper we study properties of the set-valued map

$$P_M\colon B_\infty(X) \ni C \to P_M(C) \in M \cap B(X).$$

More precisely, in Section 2 we establish a few basic properties of the functionals f_C and then show that asymptotic Chebyshev centers exist in a reflexive Banach space. The main result of this paper is the estimate for $\|x_C - x_D\|$ presented in Proposition 3.1 from Section 3. As immediate corollaries of this proposition, we deduce uniqueness of asymptotic Chebyshev centers and a fixed-point theorem for Banach spaces that are uniformly convex in every direction. Finally, we show that the map P_M is uniformly and Hölder continuous on bounded subsets of $B_\infty(X)$ in the case when the Banach space X is uniformly convex and q-convex, respectively. It should be noticed that our results extend the well-known results due to Garkavi [8], Amir [1], and Day *et al.* [5] for Chebyshev centers.

2. Existence of Asymptotic Chebyshev Centers

For any nonempty bounded subsets A and B of X, we define

$$\rho(A, B) = \inf_{x \in A} \sup_{z \in B} \|x - z\|.$$

The closed convex hull of A will be denoted by $\overline{\text{co}}(A)$. Now we establish a few properties of the functionals $f_C\colon B_\infty(X) \to [0, \infty)$ and the asymptotic Chebyshev centers x_C defined by formulae (1.1), (1.2), which will be used throughout the paper.

LEMMA 2.1. *For any* $C=\{C_n\}$ *and* $D=\{D_n\}$ *in* $B_\infty(X)$ *the following statements are satisfied*:

(a) f_C *is a nonnegative convex functional.*

(b) $|f_C(x)-f_C(y)| \leqslant \|x-y\|$ *for all* x, y *in* X.

(c) $f_C(x) \to \infty$ *as* $\|x\| \to \infty$.

(d) $|f_C(x)-f_D(x)| \leqslant H(C, D)$ *for every* x *in* X.

(e) $f_C=f_G$, *where* $G=\{\overline{\mathrm{co}}(C_n)\} \in B_\infty(X)$.

(f) $f_C(x_D) \leqslant H(C, D)+\rho(M, E)$, *where* $E=\bigcup D_n$.

Proof. We shall omit the straightforward proofs of (a)–(e). For (f), we apply (d) to get

$$f_C(x_D) \leqslant |f_C(x_D)-f_D(x_D)|+f_D(x_D) \leqslant H(C, D)+f_D(x_D).$$

This in conjunction with the fact that

$$f_D(x_D) \leqslant \inf_{x \in M} \limsup_{n \to \infty} \sup_{z \in E} \|x-z\| = \rho(M, E).$$

completes the proof. ∎

In view of the first three statements (a)–(c) given in Lemma 2.1, we can apply Theorem 5.1.3 [14, p. 138] in order to show that the functional f_C attains its minimum x_C on any nonempty closed and convex subset M of a reflexive Banach space X. In other words, we have

THEOREM 2.1. *Let* M *be a nonempty closed convex subset of a reflexive Banach space* X. *Then there exists an asymptotic Chebyshev center* x_C *of each element* $C \in B_\infty(X)$ *with respect to* M.

On the other hand, if the space X is not reflexive then there exist a nonempty closed convex subset M of X and an element $C \in B_\infty(X)$ such that the set $P_M(C)$ of asymptotic Chebyshev centers is empty. This is a direct consequence of Corollary 2.4 [17, p. 99] and remark (i) from Section 1.

3. UNIFORM CONTINUITY OF THE MAP P_M

Following Amir [1] we define the modulus of convexity $\delta_N\colon [0, 2] \to [0, 1]$ of a normed space X with respect to its subspace $N \neq \{0\}$ by the formula

$$\delta_N(\varepsilon)=\inf\left\{1-\left\|\frac{x+y}{2}\right\| : x, y \in X,\ \|x\|=\|y\|=1,\ \|x-y\|=\varepsilon,\ x-y \in N\right\}.$$

Note that if $N=X$, then δ_X is the classical modulus of convexity of X. In the definition of $\delta_N(\varepsilon)$ we can as well take the infimum over all elements $x, y \in X$ with $\|x\|, \|y\| \leqslant 1$, $\|x-y\| \geqslant \varepsilon$, and $x-y \in N$. In particular, this implies that

$$\left\|\frac{x+y}{2}\right\| \leqslant r\left[1-\delta_N\left(\frac{\|x-y\|}{r}\right)\right] \tag{3.1}$$

for all $x, y \in X$ and $r>0$, whenever $\|x\|, \|y\| \leqslant r$ and $x-y \in N$. Moreover, we have

$$\delta_N(\eta)/\eta \leqslant \delta_N(\varepsilon)/\varepsilon), \tag{3.2}$$

whenever $0<\eta \leqslant \varepsilon \leqslant 2$. In order to verify this inequality and the former statement we can repeat mutatis mutandis the proofs given in [12, pp. 60 and 66 (Lemma 1.e.8)] for the case $N=X$.

PROPOSITION 3.1. *If $x_C, x_D \in M$ are asymptotic Chebyshev centers of $C, D \in B_\infty(X)$ with respect to a nonempty closed convex subset M of a normed space X then the inequality*

$$r\delta_N\left(\frac{\|x_C-x_D\|}{r}\right) \leqslant H(C, D)$$

holds for all positive r and subspaces N of X such that

$$r \geqslant \max\{f_C(x_C), f_D(x_D)\} + H(C, D) \qquad \text{and} \qquad x_C - x_D \in N.$$

Proof. Without loss of generality we may suppose that

$$\alpha_C = f_C(x_C) \geqslant f_D(x_D) = \alpha_D.$$

By (1.1), (1.2) we conclude that, for each positive ε, there exists an integer $k(\varepsilon)$ such that

$$\alpha_C \geqslant \sup_{z \in C_n} \|x_C - z\| - \varepsilon \qquad \text{and} \qquad \alpha_D \geqslant \sup_{z \in D_n} \|x_D - z\| - \varepsilon, \tag{3.3}$$

whenever $n \geqslant k(\varepsilon)$. Moreover, using the convexity of M we get

$$\alpha_C = \inf_{x \in M} f_C(x) \leqslant f_C\left(\frac{x_C+x_D}{2}\right) \leqslant \sup_{z \in C_n}\left\|\frac{x_C+x_D}{2} - z\right\| + \varepsilon$$

for infinitely many $n \geqslant k(\varepsilon)$. Therefore, for these n there exist points $z_{n,\varepsilon} \in C_n$ such that

$$\alpha_C \leqslant \left\|\frac{x_C+x_D}{2} - z_{n,\varepsilon}\right\| + 2\varepsilon. \tag{3.4}$$

Clearly, by (3.3) we have

$$\|x_C - z_{n,\varepsilon}\| \leqslant \alpha_C + \varepsilon. \tag{3.5}$$

Next, in view of the definition of distance, one can choose points $\hat{z}_{n,\varepsilon}$ in D_n for which

$$\operatorname{dist}(z_{n,\varepsilon}, D_n) \geqslant \|z_{n,\varepsilon} - \hat{z}_{n,\varepsilon}\| - \varepsilon.$$

Hence by (3.3) we obtain

$$\begin{aligned}\|x_D - z_{n,\varepsilon}\| &\leqslant \|x_D - \hat{z}_{n,\varepsilon}\| + \|\hat{z}_{n,\varepsilon} - z_{n,\varepsilon}\| \\ &\leqslant \sup_{z \in D_n} \|x_D - z\| + \operatorname{dist}(z_{n,\varepsilon}, D_n) + \varepsilon \\ &\leqslant \sup_{z \in D_n} \|x_D - z\| + h(C_n, D_n) + \varepsilon \\ &\leqslant \alpha_D + h(C_n, D_n) + 2\varepsilon \leqslant \alpha_C + H(C, D) + 2\varepsilon. \end{aligned} \tag{3.6}$$

Now, let N be a subspace of X which includes the element $x_C - x_D$. Then applying (3.4) and (3.1) we get

$$\begin{aligned}\alpha_C - 2\varepsilon &\leqslant \left\| \frac{(x_C - z_{n,\varepsilon}) + (x_D - z_{n,\varepsilon})}{2} \right\| \\ &\leqslant s\left[1 - \delta_N\left(\frac{\|x_C - x_D\|}{s}\right)\right],\end{aligned} \tag{3.7}$$

where s is a positive number such that

$$s \geqslant \max\{\|x_C - z_{n,\varepsilon}\|, \|x_D - z_{n,\varepsilon}\|\}.$$

On the other hand, by (3.5), (3.6) we have

$$\max\{\|x_C - z_{n,\varepsilon}\|, \|x_D - z_{n,\varepsilon}\|\} \leqslant \alpha_C + H(C, D) + 2\varepsilon \leqslant r + 2\varepsilon$$

for any positive r defined as in the proposition. Hence inequalities (3.2) and (3.7) with $s := \alpha_C + H(C, D) + 2\varepsilon$ yield

$$\begin{aligned}(r + 2\varepsilon)\, \delta_N\left(\frac{\|x_C - x_D\|}{r + 2\varepsilon}\right) &\leqslant s\, \delta_N\left(\frac{\|x_C - x_D\|}{s}\right) \\ &\leqslant \alpha_C + H(C, D) + 2\varepsilon - \alpha_C + 2\varepsilon = H(C, D) + 4\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can pass to the limit $\varepsilon \to 0$ in order to get the desired inequality. ∎

This proposition can be easily used to establish the uniqueness of asymptotic Chebyshev centers. For this purpose, we recall [8] that a space X is *uniformly convex in every direction* if $\delta_N(\varepsilon) > 0$ for all one-dimensional subspaces N of X and $0 < \varepsilon \leqslant 2$. Note that every separable Banach space can be equivalently renormed to be uniformly convex in every direction [20].

THEOREM 3.1. *Let M be a nonempty closed convex subset of a uniformly convex, in every direction, space X. Then there exists at most one asymptotic Chebyshev center x_C of any element $C \in B_\infty(X)$ with respect to M.*

Proof. Suppose that x_C, $x_D \in P_M(C)$ and $x_C \neq x_D$. Then by Proposition 3.1 we have

$$0 \leqslant r\delta_N\left(\frac{\|x_C - x_D\|}{r}\right) \leqslant H(C, C) = 0$$

for $N = \operatorname{span}\{x_C - x_D\}$ and sufficiently large $r > 0$. Hence X is not uniformly convex in the direction $x_C - x_D$. ∎

It should be noticed that Theorem 3.1 was proved by Garkavi [8] for Chebyshev centers and by Day *et al.* [5, Theorem 5] for a class of relative Chebyshev centers. Moreover, by Garkavi's theorem [8, Theorem 6] and remark (ii) from Section 1 it follows that Theorem 3.1 is no longer true for normed spaces X which are not uniformly convex in every direction. It is interesting that the above results can be readily applied to prove a fixed-point theorem for a *nonexpansive self-map* of M, i.e., a map $T: M \to M$ such that

$$\|Tx - Ty\| \leqslant \|x - y\|$$

for all x, y in M. More precisely, let $F(T)$ be the *fixed-point set* of T defined by

$$F(T) = \{x \in M: Tx = x\}.$$

Moreover, denote by $A(T)$ the *asymptotic center set* of T in M which consists of asymptotic (Chebyshev) centers $x_C \in M$ of all bounded sequences $C = \{C_n\}$ with respect to M, where $C_n = T^n z$ $(n = 0, 1, \ldots)$ and z is an arbitrary element of M. Then the following theorem holds (cf. [9, Theorem 5.1, p. 22]), in which the existence part is essentially due to Zizler [20].

THEOREM 3.2. *Let T be a nonexpansive self-map of a nonempty bounded closed convex subset M of a uniformly convex in every direction reflexive Banach space X. Then it has a fixed point and $F(T) = A(T)$.*

Proof. By Theorems 2.1 and 3.1 there exists the unique asymptotic center $x_C \in M$ of $C = \{T^n z\}$ with respect to M, where z is an arbitrary fixed element of M. Since

$$\|Tx_C - T^n z\| \leqslant \|x_C - T^{n-1} z\|$$

for all $n \geqslant 1$, we obtain $f_C(Tx_C) \leqslant f_C(x_C)$ after passing to the limit superior with $n \to \infty$. Hence by the uniqueness of x_C we get $Tx_C = x_C$. Thus $\varnothing \neq A(T) \subset F(T)$. Conversely, if $z \in F(T)$ then $C_n := T^n z = z$ for all n. Consequently, we have $f_C(x) = \|x - z\|$, and so $z = x_C \in A(T)$. ∎

Clearly, if a Banach space X is uniformly convex (i.e., if $\delta_X(\varepsilon) > 0$ for $0 < \varepsilon \leqslant 2$) then Theorems 2.1–3.2 and Proposition 2.1 with $N = X$ are true. However, in this case one can prove additionally the following theorem which has been proved by Amir [1] for Chebyshev centers.

THEOREM 3.3. *Let M be a nonempty closed convex subset of a uniformly convex Banach space X. Then the single-valued map $P_M: B_\infty(X) \to M$ is uniformly continuous on every bounded subset of the metric space $B_\infty(X)$.*

Proof. A uniformly convex space is reflexive and uniformly convex in every direction. Thus Theorems 2.1 and 3.1 imply that the set $P_M(C) = \{x_C\}$ is singleton for every $C = \{C_n\} \in B_\infty(X)$. Now, if Y is a bounded subset of $B_\infty(X)$ then

$$K := \sup_{C, D \in Y} \left[\max\left\{\rho\left(M, \bigcup C_n\right), \rho\left(M, \bigcup D_n\right)\right\} + H(C, D)\right] < \infty. \tag{3.8}$$

Moreover, by Lemma 2.1(f) we have

$$\max\{f_C(x_C), f_D(x_D)\} + H(C, D) \leqslant K$$

for all C, D in Y. This in conjunction with Proposition 3.1 yields

$$K\,\delta_X\left(\frac{\|x_C - x_D\|}{K}\right) \leqslant H(C, D)$$

for all C, D in Y. Since the modulus of convexity of X is an increasing continuous function and $\delta_X(0) = 0$, it follows that the modulus of continuity $\omega_Y(P_M; \varepsilon)$ of map $P_M|_Y$ satisfies the estimate

$$\omega_Y(P_M; \varepsilon) := \sup\{\|x_C - x_D\| : C, D \in Y, H(C, D) \leqslant \varepsilon\} \leqslant K\,\delta_X^{-1}\left(\frac{\varepsilon}{K}\right).$$

Thus $\omega_Y(P_M; \varepsilon) \to 0$ as $\varepsilon \to 0$, which completes the proof. ∎

Finally, by the Amir theorem [1, Theorem 5] it follows that Theorem 3.3 is false for some nonempty bounded closed convex subset in each Banach space which is not uniformly convex.

4. Hölder Continuity of P_M

We first recall that a normed space X is said to be *q-convex* for some $q \geqslant 2$ [16] if there exists a constant $d > 0$ such that

$$\left\| \frac{x+y}{2} \right\|^q \leqslant \frac{1}{2} (\|x\|^q + \|y\|^q) - \frac{d}{2} \|x-y\|^q \tag{4.1}$$

for all x, y in X. Clearly, a q-convex space X is uniformly convex. On the other hand, by the Pisier theorem [15] it follows that each super-reflexive (in particular, uniformly convex) Banach space X can be equivalently renormed to be q-convex for some $q \geqslant 2$. Next, in view of the Clarkson and Meir inequalities (see [4, Theorem 2] and [13, Inequality 2.3]), the spaces L_p $(1 < p < \infty)$ are q-convex with $q = \max(2, p)$ and the constant d is equal to

$$d = \begin{cases} p(p-1)/4, & \text{if } 1 < p \leqslant 2, \\ 2^{1-p}, & \text{if } p \geqslant 2. \end{cases}$$

The same results hold also for the Sobolew and Hardy spaces [18, 19]. For such spaces the functional f_C defined by (1.1) has the following nice property.

Lemma 4.1. *If X is a q-convex space for some $q \geqslant 2$ and x_C is an asymptotic Chebyshew center of $C \in B_\infty(X)$ with respect to a closed convex subset M of X, then the inequality*

$$[f_C(x_C)]^q \leqslant [f_C(x)]^q - d \|x_C - x\|^q$$

holds for all x in M.

Proof. Since M is a convex set, it follows from (1.2) that

$$f_C(x_C) \leqslant f_C\left(\frac{x_C + x}{2}\right)$$

for all x in M. This in conjunction with (4.1) yields

$$[f_C(x_C)]^q \leqslant \left[f_C\left(\frac{x_C+x}{2}\right)\right]^q$$

$$\leqslant \limsup_{n\to\infty} \sup_{z\in C_n} \tfrac{1}{2}(\|x_C-z\|^q + \|x-z\|^q - d\,\|x_C-x\|^q)$$

$$\leqslant \tfrac{1}{2}([f_C(x_C)]^q + [f_C(x)]^q - d\,\|x_C-x\|^q),$$

which completes the proof. ▮

This lemma can be applied to show Hölder continuity with exponent $1/q$ of the map P_M.

THEOREM 4.1. *Let M be a nonempty closed convex subset of a q-convex Banach space X, and let Y be a bounded subset of the metric space $B_\infty(X)$. Then the single-valued map $P_M\colon B_\infty(X)\to M$ satisfies the Hölder condition*

$$\|P_M(C)-P_M(D)\| \leqslant (q/d)^{1/q}\, K^{1-1/q}(H(C,D))^{1/q}$$

for all C, D in Y, where the positive constant K is defined as in (3.8).

Proof. By Theorems 2.1 and 3.1 the set $P_M(C)$ is a singleton $x_C\in M$ for every $C\in B_\infty(X)$. Applying Lemma 4.1 twice we obtain

$$d\,\|x_C-x_D\|^q \leqslant \tfrac{1}{2}([f_C(x_D)]^q - [f_D(x_D)]^q) + \tfrac{1}{2}([f_D(x_C)]^q - [f_C(x_C)]^q)$$

for all C, D in Y. Next, we use the well-known inequality

$$|t^q-s^q| \leqslant qr^{q-1}\,|t-s|; \qquad 0\leqslant t, s\leqslant r,$$

and Lemma 2.1(d) in order to get the inequality

$$d\,\|x_C-x_D\|^q \leqslant \tfrac{1}{2}qr^{q-1}(|f_C(x_D)-f_D(x_D)| + |f_D(x_C)-f_C(x_C)|)$$

$$\leqslant qr^{q-1}H(C,D)$$

with $r := \max\{f_C(x_D), f_D(x_C)\}$. Finally, in view of Lemma 2.1(f) and (3.8) we have $r=r(C,D)\leqslant K$ for all C, D in Y. ▮

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